

BIRATIONAL UNBOUNDEDNESS OF \mathbb{Q} -FANO THREEFOLDS

JIAYUAN LIN

ABSTRACT. We prove that the family of \mathbb{Q} -Fano threefolds with Picard number one is birationally unbounded.

1. Introduction.

We work over an algebraically closed field of characteristic zero.

Definition 1.1. *Let X be a projective variety, X is said to be a \mathbb{Q} -Fano variety, if*

- (1) *X has \mathbb{Q} -factorial log terminal singularities, and*
- (2) *$-K_X$ is ample.*

The reader is referred to [14] for the standard definitions of higher-dimensional geometry, such as log terminal. The main goal of this paper is to prove the following main theorem.

Theorem 1.2. *The family of \mathbb{Q} -Fano threefolds with Picard number one is birationally unbounded.*

For the reader's convenience we state the definition of birationally bounded which is a modification of the definition of bounded given by V. Alexeev in [1].

Definition 1.3. *A class of varieties \mathcal{B} is birationally bounded if there exists a morphism $f : \mathcal{X} \rightarrow \mathcal{S}$ between two varieties such that every variety in \mathcal{B} is birational to one of the geometric fibres of f .*

Note that we do not require that every geometric fibre of f is birational to a variety in \mathcal{B} , nor do we require that every geometric fibre of f is birational to a unique variety in \mathcal{B} .

\mathbb{Q} -Fano varieties play an important role in modern birational algebraic geometry. In the Minimal Model Program (MMP) we have to allow varieties with certain mild singularities. \mathbb{Q} -Fano varieties appear naturally as one of the final results of running the log MMP. The following is an interesting and motivating conjecture proposed independently by A. Borisov [2] and V. Alexeev [1].

Conjecture 1.4 (A.Borisov-V.Alexeev). *Fix $\epsilon > 0$. Then the family of all \mathbb{Q} -Fano varieties of a given dimension with log discrepancy greater than ϵ is bounded.*

Remark. One cannot remove ϵ from the hypothesis. Take, for example, the cone over a rational curve of degree d . For every d the corresponding cone has log discrepancy $\frac{2}{d}$. These cones are all \mathbb{Q} -Fano surfaces (also known as log Del Pezzo surfaces) but they form an unbounded family.

As well as being an interesting conjecture in its own right, Conjecture 1.4 and many similar conjectures play a pivotal role in an inductive approach to higher dimensional geometry. Conjecture 1.4 also has important applications to the Sarkisov program.

For these reasons there has been a considerable amount of work on this conjecture. A.Nadel [16] and F. Campana [5] proved the boundedness of smooth Fano varieties with Picard number one. J. Kollár, Y. Miyaoka and S. Mori [11] proved the boundedness of smooth Fano varieties with arbitrary Picard number in every dimension. The case of \mathbb{Q} -Fano threefolds with Picard number one and terminal singularities is due to Y. Kawamata [9]. Z. Ran and H. Clemens [18] proved a boundedness theorem for Fano unipolar n -dimensional varieties. V. Alexeev [1] and V.V. Nikulin [17] proved the above conjecture in dimension two. A. Borisov and L. Borisov [4] gave a proof of Conjecture 1.4 in the toric case. A. Borisov [2], [3] proved that \mathbb{Q} -Fano threefolds of given index are bounded. J. Kollár, Y. Miyaoka, S. Mori and H. Takagi [12] proved that all \mathbb{Q} -Fano threefolds with canonical singularities are bounded. Recently J. McKernan [15] proved boundedness for log terminal Fano pairs of bounded index. Despite this, Conjecture 1.4 is unresolved, even in dimension three.

Consider cones over rational curves of degree d as above. Even though they form an unbounded family, all of them are birational to \mathbb{P}^2 . More generally any quotient of \mathbb{P}^2 by a finite group G is a log Del Pezzo surface of Picard number one. The classification of log Del Pezzo surfaces has attracted considerable interest. V. Alexeev and V.V. Nikulin classified log Del Pezzo surfaces under certain conditions on the indices of the singularities. One of the key steps to prove that the family of smooth Fano varieties is bounded is to bound the top self-intersection of $-K_X$. Once one allows singularities life is not so simple. In fact S. Keel and J. McKernan [10] proved that the set

$$\{ K_S^2 \mid S \text{ is a toric surface of Picard number one} \}$$

is dense in \mathbb{R}_+ . They classified all but a bounded family of Picard number one log Del Pezzo surfaces. Although there are very many log Del Pezzo surfaces up to isomorphism, all of them are rational. Thus the family of log Del Pezzo surfaces is birationally bounded and in fact birationally bounded is considerably weaker than bounded.

It has been speculated that the same is true in the higher dimensional case, that is to say it has been speculated that n -dimensional \mathbb{Q} -Fano varieties of Picard number one are birationally bounded. We find this is not the case. As a consequence, one cannot remove ϵ from the hypothesis of Conjecture 1.4, even if we replace bounded by birationally bounded.

It is also interesting to note that we cannot drop the condition that the singularities are log terminal even in the case of surfaces. Consider the family of surfaces with ample anticanonical divisor and Picard number one. This family is birationally unbounded. For example take a cone over a curve of genus g and degree greater than $2g - 2$.

The idea of the proof of Theorem 1.2 is to look at conic bundles contained in \mathbb{P}^2 -bundles over \mathbb{P}^2 , obtained as the blow ups of cones over Veronese embeddings of \mathbb{P}^2 . Varying the degree of the embedding gives an infinite family and, using deep results of Sarkisov concerning the classification of conic bundles, we prove that the conic bundles so constructed are not birational to each other. Finally we use the MMP and some results of Iskovskih [8], suitably modified, to prove that these conic bundles are birationally unbounded.

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In order to prove Theorem 1.2, we need some preliminary results.

2. Some examples of rigid conic bundles.

Definition 2.1. A projective variety X is said to be a conic bundle, if there is a morphism $\pi : X \rightarrow S$, where S is an irreducible projective variety and the geometric generic fibre is a smooth rational curve.

If there exists a rank 3 locally free sheaf \mathcal{E} on S such that π factors into a closed embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ as the zero locus of a global section v of $\text{Sym}^2 \mathcal{E}$ and the natural projection $\mathbb{P}(\mathcal{E}) \rightarrow S$, then X is a classic conic bundle. The zero locus $\Delta(v)$ of v is called the discriminant locus.

Note that the discriminant locus $\Delta(v)$ of a classic conic bundle is a divisor and that as a set it parametrises the singular fibres. We adopt these properties of the discriminant locus as the definition of the discriminant locus for an arbitrary conic bundle.

Definition 2.2. Let $\pi : X \rightarrow S$ be a conic bundle. Then the discriminant locus $\Delta(\pi)$ is defined to be the codimension one part of $\pi(R)$, where R is the locus in X where π is not smooth.

Example. Let S be any smooth surface and let X be the ordinary blow up of $S \times \mathbb{P}^1$ along any point. Then $X \rightarrow S$ is a conic bundle. On the other hand there is only one singular fibre. Thus even though the singular locus is non-empty, the discriminant locus is empty by our definition.

Definition 2.3. A conic bundle $\pi : X \rightarrow S$ is called standard if

- (1) X and S are smooth projective varieties,
- (2) π is a flat morphism,
- (3) $\rho(X) = \rho(S) + 1$, and
- (4) $-K_X$ is relative ample.

Definition 2.4. Assume X has only \mathbb{Q} -factorial terminal singularities. A Mori fibre space is an extremal contraction $f : X \rightarrow S$ of fibre type. In other words $f_*\mathcal{O}_X = \mathcal{O}_S$ and

- (1) $-K_X$ is relatively ample for f ,
- (2) $\rho(X) = \rho(S) + 1$,
- (3) $\dim S < \dim X$.

Note that a Mori fibre space of relative dimension one is automatically a conic bundle.

Definition 2.5. Let $X \rightarrow S$ and $X' \rightarrow S'$ be Mori fibre spaces. A birational map $f : X \dashrightarrow X'$ is square if it induces a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

where g is a birational map and f induces an isomorphism of the generic fibres.

We recall a result of Corti [6, Theorem 4.2], which generalises a deep result of Sarkisov [19], in the following theorem.

Theorem 2.6. *Let $X \rightarrow S$ be a standard conic bundle. Let $\Delta \subset S$ denote the discriminant curve of the conic bundle, and assume that $4K_S + \Delta$ is quasi-effective. If $X' \rightarrow S'$ is another Mori fibre space, then every birational map $\varphi : X \dashrightarrow X'$ is square.*

Recall that a \mathbb{Q} -divisor is called quasi-effective if it is a limit of effective divisors in $N^1(S)$.

The following Lemma is a minor modification of Iskovskih [8], Lemma 3 and Lemma 4. The only difference is that we have a birational map between S and S' instead of a birational morphism.

Lemma 2.7. *Let $\pi : X \longrightarrow S$ be a standard conic bundle over smooth rational surface S . Let S' be a rational surface and $\pi' : X' \longrightarrow S'$ a Mori fibre space with one dimensional fibres which is square birational to $\pi : X \longrightarrow S$, that is, there is a commutative diagram*

$$\begin{array}{ccc} X' & \overset{f}{\dashrightarrow} & X \\ \pi' \downarrow & & \downarrow \pi \\ S' & \overset{g}{\dashrightarrow} & S \end{array}$$

where f and g are birational.

- (1) *If $\pi' : X' \longrightarrow S'$ is also a standard conic bundle, then the arithmetic genera of the discriminant curves of these two standard conic bundles are equal.*
- (2) *The arithmetic genus of the discriminant curve of $\pi' : X' \longrightarrow S'$ is at least the arithmetic genus of discriminant curve of the standard conic bundle $\pi : X \longrightarrow S$.*

Proof. By elimination of indeterminacy, we know that there is a surface \bar{S} and a commutative diagram

$$\begin{array}{ccc} & \bar{S} & \\ \psi \swarrow & & \searrow \varphi \\ S' & \overset{g}{\dashrightarrow} & S \end{array}$$

where the morphisms ψ and φ are compositions of smooth blow ups. Applying Sarkisov [19] Proposition 2.4 to $\varphi : \bar{S} \longrightarrow S$, we obtain a standard conic bundle $\bar{\pi} : \bar{X} \longrightarrow \bar{S}$ such that the diagram

$$\begin{array}{ccc} \bar{X} & \dashrightarrow & X \\ \bar{\pi} \downarrow & & \downarrow \pi \\ \bar{S} & \xrightarrow{\varphi} & S \end{array}$$

is commutative.

By Iskovskikh [8] Lemma 4, we know that the arithmetic genera of the discriminant curves of these two standard conic bundles $\bar{\pi} : \bar{X} \longrightarrow \bar{S}$ and $\pi : X \longrightarrow S$ are equal. Now the diagram

$$\begin{array}{ccc}
\bar{X} & \dashrightarrow & X' \\
\bar{\pi} \downarrow & & \downarrow \pi' \\
\bar{S} & \xrightarrow{\psi} & S'
\end{array} \tag{\dagger}$$

is also commutative (where the birational map $\bar{X} \dashrightarrow X'$ is the composition of $\bar{X} \rightarrow X$ and the inverse of f). The same argument shows that the arithmetic genera of the discriminant curves of the standard conic bundles $\bar{\pi} : \bar{X} \rightarrow \bar{S}$ and $\pi' : X' \rightarrow S'$ are equal. This completes the proof of part (1).

Now we follow the proof of Iskovskikh [8] Lemma 3 to prove that the arithmetic genus of the discriminant curve $C' \subset S'$ of the conic bundle $\pi' : X' \rightarrow S'$ is at least the arithmetic genus of the discriminant curve $\bar{C} \subset \bar{S}$ of the standard conic bundle $\bar{\pi} : \bar{X} \rightarrow \bar{S}$.

Let $\mathcal{I}_{\bar{C}}$ be the ideal sheaf of \bar{C} in \bar{S} . Then we have an exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{\bar{C}} \rightarrow \mathcal{O}_{\bar{S}} \rightarrow \mathcal{O}_{\bar{C}} \rightarrow 0.$$

If we push this exact sequence down to S' , then we get a long exact sequence, part of which will be

$$0 = R^1\psi_*(\mathcal{O}_{\bar{S}}) \rightarrow R^1\psi_*(\mathcal{O}_{\bar{C}}) \rightarrow R^2\psi_*(\mathcal{I}_{\bar{C}}) = 0.$$

We have zero on the right-hand side because the fibres of ψ are at most one dimensional and also on the left-hand side because the singularities of \bar{S} and S' are log terminal, whence rational. The morphism ψ induces a contraction morphism $\psi|_{\bar{C}} : \bar{C} \rightarrow C'$, which is an isomorphism over a dense open subset of C' , as (\dagger) is a commutative diagram. Thus the Leray spectral sequence for $\psi|_{\bar{C}}$ degenerates at the E_2 -level and in particular $h^i(\mathcal{O}_{\bar{C}}) = h^i(\psi_*\mathcal{O}_{\bar{C}})$.

The exact sequence

$$0 \rightarrow \mathcal{O}_{C'} \rightarrow \psi_*\mathcal{O}_{\bar{C}} \rightarrow \mathcal{K} \rightarrow 0,$$

where \mathcal{K} is defined by exactness, yields an exact cohomology sequence

$$0 \rightarrow H^0(\mathcal{K}) \rightarrow H^1(\mathcal{O}_{C'}) \rightarrow H^1(\psi_*\mathcal{O}_{\bar{C}}) \rightarrow 0.$$

This implies that $h^1(\mathcal{O}_{\bar{C}}) = h^1(\psi_*\mathcal{O}_{\bar{C}}) \leq h^1(\mathcal{O}_{C'})$. Combining with the first part of this Lemma gives the proof of part (2). \square

We now want to focus on a sequence of conic bundles over \mathbb{P}^2 , first introduced by Sarkisov [19].

Lemma 2.9. *Let $\pi : X \longrightarrow \mathbb{P}^2$ be the projectivisation of the vector bundle $E = \mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ over \mathbb{P}^2 . Let L be the tautological line bundle on $\mathbb{P}(E)$ so that $\pi_* L = E$ and let F be the pullback of the generator of $\text{Pic}(\mathbb{P}^2)$.*

Then $L^{\otimes 2} \otimes F$ is ample and base-point free and a generic divisor V_k from the linear system $|L^{\otimes 2} \otimes F|$ on X is a standard conic bundle over \mathbb{P}^2 whose discriminant curve is smooth of degree $2k + 3$.

Proof. The projectivisation X of the vector bundle $E = \mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ over \mathbb{P}^2 has Picard number two and its cone of curves is generated by the class of a line contained in a fibre and the class of a line contained in the unique section with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-k) \oplus \mathcal{O}_{\mathbb{P}^2}(-k)$. The line bundle $L^{\otimes 2} \otimes F$ is positive on both of these rays and so $L^{\otimes 2} \otimes F$ is certainly ample. On the other hand X is toric (indeed it is the projectivisation of a direct sum of line bundles over a toric variety) and so $L^{\otimes 2} \otimes F$ is automatically base-point free (e.g. see Fulton [7] page 70).

Let V_k be defined by $v_k = 0$ for some $v_k \in H^0(L^{\otimes 2} \otimes F)$. Since

$$\begin{aligned} \pi_*(L^{\otimes 2} \otimes F) &= \text{Sym}^2 E \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\ &= \mathcal{O}_{\mathbb{P}^2}(2k+1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(k+1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(k+1), \end{aligned}$$

we may write

$$\pi_*(v_k) = (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{23}, 2a_{31})$$

where $a_{11} \in H^0(\mathcal{O}_{\mathbb{P}^2}(2k+1))$, $a_{22} \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a_{33} \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a_{12} \in H^0(\mathcal{O}_{\mathbb{P}^2}(k+1))$, $a_{23} \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_{31} \in H^0(\mathcal{O}_{\mathbb{P}^2}(k+1))$. It is easy to see that the discriminant curve $\Delta(V_k)$ is the vanishing locus of

$$D(\{a_{ij}\}) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in H^0(\mathcal{O}_{\mathbb{P}^2}(2k+3)) \quad (1)$$

where we set $a_{ij} = a_{ji}$. In particular $\deg \Delta(V_k) = 2k + 3$.

It remains to show that for a general choice of a_{ij} , the curve $D(\{a_{ij}\}) = 0$ given in (1) is smooth. If $a_{12} = a_{23} = a_{31} = 0$, the corresponding curve $D(\{a_{ij}\}) = 0$ is a union $C_1 \cup C_2 \cup C_3$, where $C_1 = \{a_{11} = 0\}$ is a general curve of degree $2k + 1$ and $C_i = \{a_{ii} = 0\}$, $i = 2, 3$ are two lines in general position. So the singularities of $D(\{a_{ij}\}) = 0$ are $C_i \cap C_j$. We will show that these singularities are smoothed out as v_k deforms.

Consider the family of curves given by

$$\det \begin{pmatrix} a_{11} & ta_{12} & 0 \\ ta_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = (a_{11}a_{22} - t^2a_{12}^2)a_{33} = 0.$$

When $t = 0$, we have $C_1 \cup C_2 \cup C_3$. When $t \neq 0$, we see that the singularities $C_1 \cap C_2$ are smoothed out as long as we pick $a_{12} \in H^0(\mathcal{O}_{\mathbb{P}^2}(k+1))$ not vanishing at $C_1 \cap C_2$. Similarly, one can show that the singularities $C_2 \cap C_3$ and $C_3 \cap C_1$ are smoothed out as v_k deforms. \square

Remark. The degree of the discriminant curve of V_k was already known. It is well known that the discriminant curve of a standard conic bundle is nodal. However the fact that $\Delta(V_k)$ is smooth, when V_k is general, is a new result due to Xi Chen. James M^cKernan even wonders if the following is true:

Let X be the projectivisation of a rank 3 vector bundle over a smooth surface S . Suppose that $|L^{\otimes 2} \otimes F^{\otimes k}|$ is very ample, where L restricts to $\mathcal{O}_{\mathbb{P}^2}(1)$ on a fibre. Then for generic choice of V belonging to the linear system $|L^{\otimes 2} \otimes F^{\otimes k}|$ the corresponding discriminant curve is smooth.

Lemma 2.9. *Let $k, m \geq 5$. If V_k is birational to V_m , then $k = m$.*

Proof. Assume V_k is birational to V_m and let Φ denote the birational map. As $k \geq 5$ Theorem 2.6 applies, therefore $\Phi : V_k \dashrightarrow V_m$ is square, that is Φ induces a commutative diagram

$$\begin{array}{ccc} V_k & \overset{\Phi}{\dashrightarrow} & V_m \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2. \end{array}$$

By Lemma 2.7 (1) the arithmetic genera of the discriminant curves of V_k/\mathbb{P}^2 and V_m/\mathbb{P}^2 are equal. Since the discriminant curves of V_k and V_m are smooth plane curves of degree $2k+3$ and $2m+3$, respectively, we have that $k = m$. \square

Lemma 2.10. *The standard conic bundle V_k is birational to a \mathbb{Q} -Fano threefold V'_k with Picard number one.*

Proof. Consider the k -uple Veronese embedding of \mathbb{P}^2 into \mathbb{P}^N , where $N = \frac{(k+2)(k+1)}{2} - 1$. Pick a line l skew to \mathbb{P}^N in \mathbb{P}^{N+2} . Let X'_k be the linear join of the image S_k of \mathbb{P}^2 under this embedding and the line l . Blowing up X'_k along l we obtain a fourfold X_k which is a \mathbb{P}^2 -bundle over \mathbb{P}^2 . Now X_k is isomorphic to $\mathbb{P}_{\mathbb{P}^2}(G)$, where G is a rank 3 vector bundle over \mathbb{P}^2 .

The normal bundle S_k in X_k is $\mathcal{O}_{\mathbb{P}^2}(k) \oplus \mathcal{O}_{\mathbb{P}^2}(k)$. On the other hand S_k is a section of the natural map $X_k \rightarrow \mathbb{P}^2$ and so S_k corresponds to a quotient $G \rightarrow Q = \mathcal{O}_{X_k}(a)$ of G . Let K be the kernel. Then the normal bundle of S_k in X_k is canonically isomorphic to $\text{Hom}(K, Q) = K^* \otimes Q$. It follows that K is split so that $K = \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}(c)$ for some b and c . We have $a - b = a - c = k$. Thus $b = c$. Tensoring by a line bundle, we may as well assume that

$b = c = 0$. In this case $a = k$. Thus X_k is indeed isomorphic to $\mathbb{P}_{\mathbb{P}^2}(E)$, where E is the vector bundle appearing in (2.8).

Let W_k be the image of V_k inside X'_k and let $f_k: V_k \rightarrow V'_k$ be the induced contraction, so that V'_k is the normalisation of W_k . Now f_k is birational; let E be the exceptional locus (we use the same notation as we used for the underlying vector bundle; hopefully this will not cause confusion). Then E is a divisor, which intersects the general fibre F of $V_k \rightarrow \mathbb{P}^2$ in two points.

The Picard number of X_k is clearly equal to two. Hence V_k also has Picard number two, as V_k is ample in X_k . Thus V'_k has Picard number one. In particular f_k has relative Picard number one.

As X'_k has \mathbb{Q} -factorial singularities it follows that $K_{V'_k}$ is \mathbb{Q} -Cartier. Let a be the log discrepancy of E , so that

$$K_{V_k} + E = \pi^* K_{V'_k} + aE.$$

To compute a , pick a line L inside E and intersect both sides of this equation with L ,

$$-3 = K_E \cdot L = (K_{V_k} + E) \cdot L = (\pi^* K_{V'_k} + aE) \cdot L = aE \cdot L = -ak.$$

Thus $a = \frac{3}{k}$ and V_k is log terminal, of log discrepancy $\frac{3}{k}$. Moreover it follows that

$$K_{V_k} + (1 - a')E$$

is f_k -negative, for any $a' > a$. As f_k has relative Picard number one, it follows that f_k is a step of the $K_{V_k} + (1 - a')E$ -MMP and in particular V'_k is \mathbb{Q} -factorial.

Finally we check that $-K_{V'_k}$ is ample. As V'_k has Picard number one, it is enough to check that $K_{V'_k} \cdot C < 0$ for any curve C in V'_k . Let F be a general fibre of $V_k \rightarrow \mathbb{P}^2$ and let C be the image of F in V'_k . Then $K_{V_k} \cdot F = -2$ and $E \cdot F = 2$. Thus $K_{V_k} + (1 - a)E$ is negative on F . It follows by push-forward that $K_{V'_k} \cdot C < 0$, as required. Thus V'_k is indeed a \mathbb{Q} -Fano threefold of Picard number one. \square

3. Proof of Theorem 1.2.

Let $k, m \geq 5$ from now on. Assume that the family of \mathbb{Q} -Fano threefolds with Picard number one is birationally bounded. Then there exists a parameter space B of finite type and a morphism $\pi: \mathcal{X} \rightarrow B$ such that any \mathbb{Q} -Fano threefold is birational to at least one fibre of π .

Let $S \subset B$ be the set of those points of $b \in B$ such that X_b is birational to V_k , for some k . Now by Lemma 2.10, V_k is birational to V'_k , a \mathbb{Q} -Fano threefold, and moreover by Lemma 2.9, V_k is not birational to V_m , for $m \neq k$. It follows that there are infinitely many fibers over S which are not birational to each other.

In the course of the proof of Theorem 1.2, we are going to repeatedly modify B whilst still preserving the property that there are infinitely many fibers over S which are not birational to each other. For example we are clearly free to replace B by the closure of the points corresponding to V_k for infinitely many k . The fact that there are infinitely many fibers over S which are not birational to each other implies that the set of k such that V_k is birational to X_b is infinite.

Let Z denote the closure of S . Then there is an irreducible component of Z which contains an infinite subset of S . Renaming this component B , we may assume that the set S is dense in B and that B is irreducible. Note that B has positive dimension, as S is infinite.

For the generic point $\eta \in B$, pick a resolution of \mathcal{X}_η . Replacing B by an open neighborhood of η we may assume that \mathcal{X}/B is a flat family of smooth projective three dimensional algebraic varieties over B . We recall a deep result due to Kollár and Mori, [13], which states that one can run the MMP in families.

Theorem 3.1. *Let B be a connected normal quasi-projective variety and let X/B be a flat, projective family of threefolds such that every fibre has only \mathbb{Q} -factorial terminal singularities.*

Then there is a finite, étale and Galois base change $p : B' \rightarrow B$, a flat projective family Y/B' , and a rational map $f' : X'/B' \dashrightarrow Y/B'$ such that on each fibre f' induces a birational map, each fibre of Y/B' has only \mathbb{Q} -factorial terminal singularities and either

- (a) $K_{Y/B'}$ is relatively nef, or
- (b) *There is a morphism $Y \rightarrow Z$ over B' , where $-K_Y$ is relatively ample, and $Y_{b'} \rightarrow Z_{b'}$ is extremal, for every $b' \in B'$.*

Here X' is the family over B' obtained by base change.

Proof. This is [13, Proposition 12.4.2]. \square

Applying Theorem 3.1 to our situation, possibly passing to an étale cover, we may assume that either (a) or (b) of Theorem 3.1 holds. (a) is clearly impossible, since infinitely many fibres are uniruled, and if X_b is uniruled, then K_{X_b} is certainly not nef.

Thus we may assume that there is a morphism $\pi : \mathcal{X} \rightarrow \mathcal{Z}$ over B , such that $X_b \rightarrow Z_b$ is a Mori fibre space, for all $b \in B$. Suppose that $b \in S$. As $k \geq 5$ Theorem 2.6 applies and the resulting birational map $X_b \dashrightarrow V_k$ is square. In particular for every $b \in S$, Z_b is a surface. As S is dense in B , $\mathcal{Z} \rightarrow B$ is therefore a family of surfaces. Equivalently, $\mathcal{X} \rightarrow \mathcal{Z}$ is a family of conic bundles.

Let $\Delta \subset \mathcal{Z}$ be the locus where π is not smooth. Then for every point $b \in S$, Δ_b is a curve in Z_b . Thus, possibly passing to an open subset of B , we may assume that the discriminant locus of $X_b \rightarrow Z_b$ is non-empty and equal to Δ_b . Possibly passing to an open subset of B , we

may assume that Δ is flat over B so that the arithmetic genera of the discriminant curves are constant over B .

On the other hand, the arithmetic genus of the discriminant curve of the standard conic bundle V_k over \mathbb{P}^2 is $(k+1)(2k+1)$. Suppose that $b \in S$. According to Lemma 2.7 (2), the arithmetic genus of Δ_b is at least $(k+1)(2k+1)$. As there are infinitely many fibers over S which are not birational to each other, the set of such k is infinite, and this contradicts the fact that the arithmetic genera are constant.

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